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1971 J. Phys. A: Gen. Phys. 4 73

(http://iopscience.iop.org/0022-3689/4/1/010)

J. Phys. A: Gen. Phys., 1971, Vol. 4. Printed in Great Britain

On Phythian's perturbation theory for stationary homogeneous turbulence

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MS. received 1*st April* 1970

Abstract. It is pointed out that the perturbation expansion procedure about a certain flow state requires *a priori* knowledge of the statistical distribution of the flow field. In Wyld's laminar perturbation procedure it was indeed natural by invoking Kraichnan's maximal randomness principle to assume the laminar flow to have a Gaussian distribution. On the other hand, if we attempt to develop perturbation about a turbulent flow as in Phythian's procedure, the difficulty is that the distribution of the velocity field is just as unknown as the turbulence problem itself. We shall show that Phythian's assumption of Gaussian random force for the zeroth-order equation has the effect of deriving the direct-interaction equations under the quasi-normal hypothesis.

1. Introduction

Recently, Phythian (1969) has proposed a so-called selfconsistent perturbation procedure for the stationary homogeneous turbulence problem of an incompressible fluid. The novel feature of his approach is the development of perturbation about a turbulent flow which is supposed to have the same covariance and response functions as the actual problem. To carry out such a perturbation procedure, however, necessitates *a priori* knowledge of the statistical distribution of the turbulent fluid, which Phythian has assumed Gaussian for expediency. As a result, Phythian's set of the direct-interaction equations is the one derived under the explicit assumption of quasi-normality. Let us consider here the Burgers model turbulence in a stationary and homogeneous velocity field. For a space–time box satisfying the cyclic boundary conditions, we have the following Fourier amplitude equations in the notation of Phythian (1969):

$$\mathrm{i}\omega v(k,\omega) = -\nu k^2 v(k,\omega)+f(k,\omega)+\frac{\lambda}{VT}\Sigma M(k,\omega;k_1,\omega_1,k_2,\omega_2)v(k_1,\omega_1)v(k_2,\omega_2) \tag{1}$$

where

$$M(k,\omega;k_1,\omega_1,k_2,\omega_2) = -\mathrm{i}\tfrac{1}{2}k\delta_{k_1+k_2,k}\delta_{\omega_1+\omega_2,\omega}$$

ν being the kinematic viscosity, and λ is a parameter associated with the nonlinear term. Here, $v(k,\omega)$ and $f(k,\omega)$ are the respective amplitude coefficients of the velocity field $v(x,t)$ and applied force field $f(x,t)$, when Fourier analysed in a space-time box of volume VT. We note that $v(k,\omega)$ and $f(k,\omega)$ satisfy the reality requirements and have zero ensemble averages of their moments unless both the k and ω have zero sums.

2. Wyld's theory

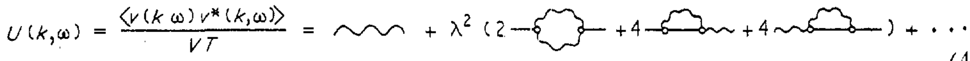
In 1961, Wyld presented the formal expansions for the covariance and response functions by a perturbation procedure which consists of two operations. The first operation is the conventional perturbation about the laminar flow. Rewrite (1) as

$$v = sf + \frac{\lambda}{VT} s\, \Sigma M v v \tag{2}$$

where $s = (\mathrm{i}\omega + \nu k^2)^{-1}$ is the laminar propagator. If we develop the laminar perturbation expansion by introducing $v = v_0 + \lambda v_1 + \ldots$ into (2), then the zeroth-order problem ($\lambda = 0$) is the laminar flow

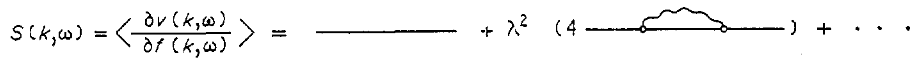
$$v_0 = sf \tag{3}$$

and the remaining v_n for $n \geqslant 1$ are defined successively in terms of v_0 only. Although there is nothing new analytically in the laminar perturbation, the introduction of statistics into the dynamics deserves a special comment. To do this in an exact way, Wyld has invoked Kraichnan's (1958) maximal randomness principle which postulates the fully developed turbulent dynamics to be as random as is possible consistent with the nonlinear dynamics of (1), but not at all dependent upon the initial and boundary conditions. Since the most random process is of Gaussian nature, Wyld requires the f to have a Gaussian distribution. Then, in view of the linearity of (3), the laminar field v_0 would have the same Gaussian distribution as f, and hence is maximally random. Under this condition, the computation of terms of arbitrary order in the laminar perturbation expansion is straightforward. We shall present here only the leading terms of the covariance



(4)

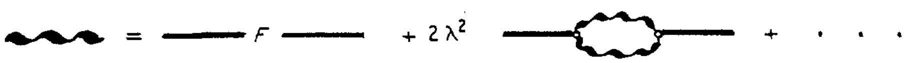
and the response function



(5)

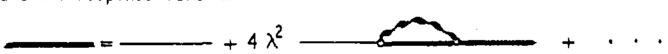
where $\langle\rangle$ is ensemble average. In the above expansions, we have used for simplicity the diagram representations: the laminar covariance $\langle v_0 v_0{}^*\rangle/VT \leftrightarrow$,† the laminar response function $s \leftrightarrow$ —, and the vertex $(VT)^{-1}\, \Sigma\, M \leftrightarrow$ o. A very important point that must be observed from (4) and (5) is this. Each term of the laminar perturbation expansions has been computed explicitly taking v_0 as Gaussian; however, the perturbation expansions as infinite sums represent a non-Gaussian process.

The second operation of Wyld's theory is the rearrangement and summation of the laminar perturbation expansions which are familiar in quantum field theory. By summing up certain reducible classes of the laminar diagrams to all orders, Wyld has obtained the consolidated expansions for the covariance



(6)

and the response function



(7)

where $U \leftrightarrow$, $S \leftrightarrow$, and $F = \langle ff^*\rangle/VT$. We observe that the expansions

† Note: here stands for the *thin* wavy line in equations (4) and (5).

(6) and (7) involve the turbulent covariance and response functions themselves, and truncation of the terms of order higher than $O(\lambda^2)$ yields the direct-interaction approximation. It must be noted that the irreducible second-order laminar diagrams of (4) and (5) reappear as the respective direct-interaction terms of (6) and (7).

3. Phythian's theory

To circumvent Wyld's second operation, which not only is tedious but also requires a good deal of intuition, Phythian (1969) has devised a formal technique for deriving the direct-interaction equations more efficiently without necessitating tedious summations of the laminar perturbation expansions. We see that his procedure also involves two operations, intertwined and simultaneous for each order of approximation. The first operation is the perturbation about a certain turbulent flow state and the second is the elimination of the reducible kind of turbulent diagrams from the perturbation expansions for the covariance and response functions. In Phythian's theory, we look for the possibility of approximating certain of the actual turbulent dynamics by a linear equation of the form

$$\mathrm{i}\omega v(k, \omega) = \alpha(k, \omega) v(k, \omega) + g(k, \omega) \tag{8}$$

where the functions g (random) and α (non-random) are yet unspecified. The covariance and response functions of (8) are given very simply by

$$U = \frac{\langle gg^* \rangle}{VT(\Omega\Omega^*)} \quad \text{and} \quad S = \left\langle \frac{\partial v}{\partial g} \right\rangle = \Omega^{-1} \tag{9}$$

where $\Omega = \mathrm{i}\omega - \alpha$. Although (9) are the exact statistical functions for (8), they are not in any way related to the actual problem (1). Therefore, the essence of Phythian's approach is to relate (8) and (1) by choosing α and g in such a way that (8) can best approximate the dynamics of (1) in a statistical sense. Then, there is a chance for expressing the U and S of (1) by the simple prescription (9).

To this end, we assume the following expansions

$$\begin{aligned} \alpha(k, \omega) &= -\nu k^2 - \sum_{n=1}^{\infty} \lambda^n R_n(k, \omega) \\ g(k, \omega) &= f(k, \omega) - \sum_{n=1}^{\infty} \lambda^n e_n(k, \omega) \end{aligned} \tag{10}$$

which ensure the correct laminar behaviours as $\lambda \to 0$. In order to implement the evaluation of R_n and e_n, we introduce $\Sigma_{n=1}^{\infty} \lambda^n R_n v + \Sigma_{n=1}^{\infty} \lambda^n e_n$ into both sides of (1)

$$\begin{aligned} \mathrm{i}\omega v - \left(-\nu k^2 - \sum_{n=1}^{\infty} \lambda^n R_n\right) v - \left(f - \sum_{n=1}^{\infty} \lambda^n e_n\right) \\ = \sum_{n=1}^{\infty} \lambda^n R_n v + \sum_{n=1}^{\infty} \lambda^n e_n + \frac{\lambda}{VT} \Sigma M v v. \end{aligned} \tag{11}$$

Identifying the two bracket terms in the left-hand side of (11) respectively with α and g, and replacing the λ in the right-hand side by μ, we obtain the basic equation of Phythian's theory

$$\mathrm{i}\omega v - \alpha v - g = \sum_{n=1}^{\infty} \mu^n R_n v + \sum_{n=1}^{\infty} \mu^n e_n + \frac{\mu}{VT} \Sigma M v v. \tag{12}$$

Although (12) apparently involves both λ and μ, the presence of μ is immaterial because it plays the role of a dummy parameter under the selfconsistency requirement that the zeroth-order equation ($\mu = 0$) has the same covariance and response functions as the actual problem (1). It is quite tempting to introduce $v = u_0 + \mu u_1 + \dots$ into (12), and thereby develop the perturbation expansion about a turbulent flow which is required to have the same U and S as the actual problem. The difficulty however is that there is no rigorous way of introducing statistical elements into the dynamics as was done in Wyld's laminar perturbation procedure by invoking the maximal randomness principle. To overcome this difficulty, Phythian has introduced the additional assumption that g has a Gaussian distribution. Since (8) is linear, this implies that u_0 has the same Gaussian distribution as g. And, in view of the selfconsistency requirement, it further means that the turbulent field itself is also Gaussian inasmuch as U and S are concerned, i.e. the quasi-normal hypothesis.

Under the assumption of a Gaussian g, the turbulent perturbation procedure goes through exactly in parallel with Wyld's laminar perturbation procedure. Upon imposing the selfconsistency requirement that the zeroth-order problem has the same U and S as the actual problem, the R_n and e_n can be determined successively in terms of $\langle gg^* \rangle$ only. We note that the evaluation of R_n and e_n by Phythian seems to have a diagrammatic analogue of eliminating the reducible turbulent diagrams of the same order at each stage of approximation. In this way, Phythian finally obtains the same turbulent perturbation expansions for U and S as (6) and (7) but without examining the laminar diagrams of arbitrary high order as was required in Wyld's procedure. However, it must be emphasized that the quasi-normal hypothesis plays the central role in Phythian's derivation of the direct-interaction equations. Before closing, we wish to indicate an obvious connection between Phythian's basic equation (12) and the recent turbulence model equation by Kraichnan (1970). Suppose that only the R_2 and e_2 are non-vanishing in the sums of (10). Then, by letting $\lambda = 1$, (12) can be written as

$$(i\omega + \nu k^2)v + (1-\mu^2)R_2 v = f - (1-\mu^2)e_2 + \frac{\mu}{VT}\Sigma Mvv.$$

Note that this has the same general form as Kraichnan's model equation except for the external force terms.

Acknowledgments

Correspondence with Professor R. Phythian has been very helpful in clarifying certain aspects of his perturbation technique.

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